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Multiple Choice

1.(6 pts.) Evaluate the following limit:

$$\lim_{x\to 3} \frac{\sqrt{x^2+7}-4}{x-3}.$$

Solution:

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{x^2+7}-4}{x-3} &= \lim_{x\to 3} \frac{\sqrt{x^2+7}-4}{x-3} \cdot \frac{\sqrt{x^2+7}+4}{\sqrt{x^2+7}+4} \\
&= \lim_{x\to 3} \frac{(x^2+7)-16}{(x-3)(\sqrt{x^2+7}+4)} \\
&= \lim_{x\to 3} \frac{x^2-9}{(x-3)(\sqrt{x^2+7}+4)} \\
&= \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)(\sqrt{x^2+7}+4)} \\
&= \lim_{x\to 3} \frac{x+3}{\sqrt{x^2+7}+4} \\
&= \frac{3+3}{\sqrt{3^2+7}+4} \\
&= \frac{6}{8}
\end{aligned}$$

2.(6 pts.) Evaluate the following limit:

$$\lim_{x\to\, -\mathbf{2}} \frac{x^2+3x+2}{(x+2)|x+1|}.$$

Solution:
Since $|x+1| = -(x+1)$ when $x < -1$, and therefore when x is close to -2,

$$\begin{aligned}
\lim_{x\to\, -\mathbf{2}} \frac{x^2+3x+2}{(x+2)|x+1|} &= \lim_{x\to\, -\mathbf{2}} \frac{(x+2)(x+1)}{-(x+2)(x+1)} \\
&= -1
\end{aligned}$$

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3.(6 pts.) For what value a is the function f given by

$$f(x) = \begin{cases} \dfrac{x+2}{x^2+x+1} & x > 0 \\ a & x = 0 \\ \dfrac{x^2-1}{x-1} & x < 0 \end{cases}$$

continuous at $x = 0$?

Solution:
For f to be continuous, $\lim_{x\to 0} f(x) = f(0)$. It is easy to see that $f(0) = a$. Next, we want to find $\lim_{x\to 0} f(x)$. In order to do this, we need to calculate $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x+2}{x^2+x+1} = \frac{0+2}{0^2+0+1} = 2$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x^2-1}{x-1} = \lim_{x\to 0^-} \frac{(x+1)(x-1)}{x-1} = \lim_{x\to 0^-} (x+1) = 0+1 = 1$$

Since $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$, the $\lim_{x\to 0} f(x)$ does not exist. Hence, there is no value of a makes f continuous at $x = 0$.

4.(6 pts.) Find $f'(2)$, if

$$f(x) = \frac{\sqrt{x}+2}{\cos(\pi x)}.$$

Solution:

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Use the differentiation rules to find the derivative. Finding the solution is fastest if you substitute $x = 2$ before simplifying the result. Observe:

$$\begin{aligned}
f'(x) &= \frac{(\sqrt{x}+2)'\cos(\pi x) - (\sqrt{x}+2)(\cos(\pi x))'}{\cos^2 \pi x} \\
&= \frac{\dfrac{\cos(\pi x)}{2\sqrt{x}} + (\sqrt{x}+2)\pi\sin(\pi x)}{\cos^2(\pi x)} \\
f'(2) &= \frac{\dfrac{\cos(2\pi)}{2\sqrt{2}} + (\sqrt{2}+2)\pi\sin(2\pi)}{\cos^2(2\pi)} \\
&= \frac{\dfrac{1}{2\sqrt{2}} + (\sqrt{2}+2)\pi 0}{1} \\
&= \frac{1}{2\sqrt{2}}.
\end{aligned}$$

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5.(6 pts.) Find the derivative of $f(x) = \cos(\cos(x^3))$.

Solution:
We must use the chain rule twice.

$$\begin{aligned} f'(x) &= -\sin(\cos(x^3))[(-\sin(x^3))(3x^2)] \\ &= 3x^2 \sin(x^3)\sin(\cos(x^3)) \end{aligned}$$

6.(6 pts.) Compute $f'(1)$, if

$$f(x) = (x+1)^{3/2}(x^2+x+1).$$

Solution:
By the product rule, we have that

$$f'(x) = \frac{3}{2}(x+1)^{1/2}(x^2+x+1) + (x+1)^{3/2}(2x+1)$$

In particular, this implies $f'(1) = \frac{3}{2}\sqrt{2}(3)+\sqrt{2^3}(3) = \frac{9}{2}\sqrt{2}+6\sqrt{2} = \sqrt{2}\left(\frac{9}{2}+6\right) = \frac{21\sqrt{2}}{2}$

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7.(6 pts.) If $f(x) = \dfrac{1}{1-x}$, find $f^{(2)}(x)$.

Solution:

First, let's write $f(x) = (1-x)^{-1}$, then

$$\begin{aligned} f'(x) &= (-1)(1-x)^{-2}(-1) \\ &= (1-x)^{-2} \\ f^{(2)}(x) &= (-2)(1-x)^{-3}(-1) \\ &= \frac{2}{(1-x)^3} \end{aligned}$$

8.(6 pts.) Find the equation of the tangent line to the curve $y = \sqrt{x} + \dfrac{1}{x}$ at $x = 4$.

Solution:

First find y and $\dfrac{dx}{dy}$ at $x = 4$. Then use the point-slope equation of a line to find the correct answer.

When $x = 4$,

$$\begin{aligned} y &= \sqrt{4} + \frac{1}{4} \\ &= 2 + \frac{1}{4} \\ &= \frac{9}{4} \\ \frac{dy}{dx} &= \frac{1}{2\sqrt{x}} - \frac{1}{x^2}. \\ \left.\frac{dy}{dx}\right|_4 &= \frac{1}{2\sqrt{4}} - \frac{1}{4^2} \\ &= \frac{1}{4} - \frac{1}{16} \\ &= \frac{3}{16}. \end{aligned}$$

Therefore, the equation of the tangent line is

$$y - \frac{9}{4} = \frac{3}{16}(x-4).$$

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9.(6 pts.) Find y', if

$$3x^2 + x^2y + xy^2 = 1.$$

Solution:

We use implicit differentiation.

$$\begin{aligned}
3x^2 + x^2y + xy^2 = 1 &\implies 6x + (2xy + x^2(1)y') + ((1)y^2 + x(2y)y') = 0 \\
&\implies 6x + 2xy + x^2y' + y^2 + 2xyy' = 0 \\
&\implies 6x + 2xy + y^2 + y'(x^2 + 2xy) = 0 \\
&\implies y'(x^2 + 2xy) = -6x - 2xy - y^2 \\
&\implies y' = \frac{-(6x + 2xy + y^2)}{x^2 + 2xy}
\end{aligned}$$

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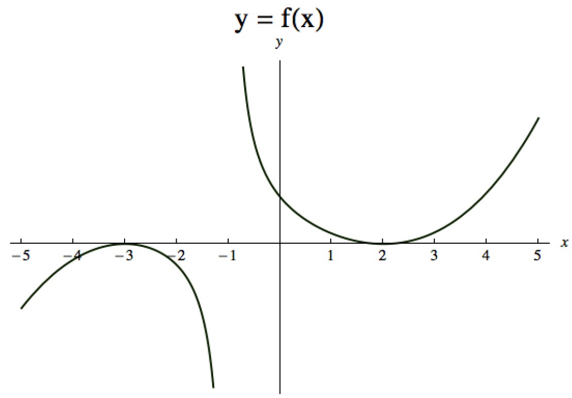
10.(6 pts.) The graph of the function $f(x)$ is shown below:



The graph of $f(x)$ has a vertical asymptote at $x = -1$.
Which of the following is the graph of $f'(x)$?
(all of the graphs below have vertical asymptotes at $x = -1$).

Solution: As the derivative to the left of $x = -1$ goes to negative infinity, and the derivative to the right of $x = -1$ goes to negative infinity, (a), (b), and (c) are incorrect. The derivative changes from positive to negative at $x = -3$, and from negative to positive at $x = 2$, so (d) is a perfectly reasonable sketch of the derivative.

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Partial Credit

You must show your work on the partial credit problems to receive credit!

11.(8 pts.) Use the Squeeze/Sandwich Theorem to find

$$\lim_{x\to 0} x^4 \sin\left(\frac{1}{x^2+2}\right).$$

Justify each step in your argument.

Solution:

First, note that $-1 \leq \sin\left(\dfrac{1}{x^2+2}\right) \leq 1$. Multiplying these inequalities through by x^4, we have

$$-x^4 \leq x^4 \sin\left(\frac{1}{x^2+2}\right) \leq x^4.$$

And $\lim_{x\to 0}(-x^4) = 0 = \lim_{x\to 0}(x^4)$. So, by the Squeeze Theorem, we obtain

$$\lim_{x\to 0} x^4 \sin\left(\frac{1}{x^2+2}\right) = 0.$$

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12.(12 pts.) Find the derivative of

$$f(x) = \frac{x}{x-2}$$

using the limit definition of the derivative.
Please include all of the details in your calculation.

Solution:

Proceed directly from the definition. Use algebraic rules to simplify the resulting limit until the expression is continuous (as a function of h) at $h = 0$, then substitute.

Observe:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\dfrac{x+h}{x+h-2} - \dfrac{x}{x-2}}{h} \\
&= \lim_{h \to 0} \frac{(x+h)(x-2) - x(x+h-2)}{h(x+h-2)(x-2)} \\
&= \lim_{h \to 0} \frac{x^2 - 2x + hx - 2h - x^2 - xh + 2x}{h(x+h-2)(x-2)} \\
&= \lim_{h \to 0} \frac{-2h}{h(x+h-2)(x-2)} \\
&= \lim_{h \to 0} \frac{-2}{(x+h-2)(x-2)} \\
&= -\frac{2}{(x-2)^2}.
\end{aligned}$$

To double check your work, you can verify that this is the same answer you get using the quotient rule.

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13.(10 pts.) Consider the following table of function values:

	$x = 1$	$x = 2$
$f(x)$	-1	3
$f'(x)$	1	2
$g(x)$	2	1
$g'(x)$	-2	-1

(Show all of your work below for credit)

(a) Find $\left(\frac{f}{g}\right)'(1)$.

Solution:
The quotient rule says

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Therefore,

$$\begin{aligned}\left(\frac{f}{g}\right)'(1) &= \frac{g(1)f'(1) - f(1)g'(1)}{(g(1))^2} \\ &= \frac{(2)(1) - (-1)(-2)}{2^2} \\ &= \frac{2-2}{4} \\ &= 0.\end{aligned}$$

(b) Find $(f \circ g)'(2)$.

Note: $(f \circ g)(x) = f(g(x))$.

Solution:
The chain rule says

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Therefore,

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$(f \circ g)'(2) = f'(g(2))g'(2) = f'(1)g'(2) =$ **(1)(-1) = (-1)**

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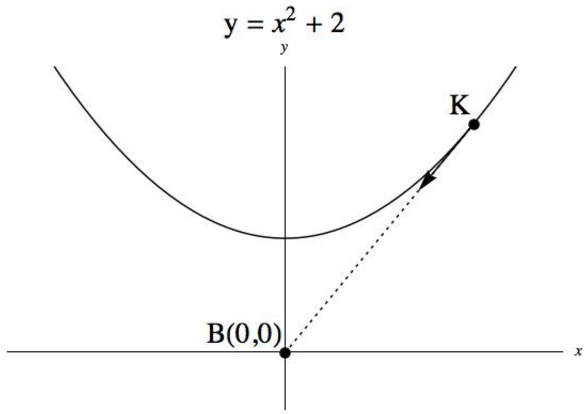
14.(10 pts.) Katniss is moving along the parabola $y = x^2+2$ shown below. As she moves, the x coordinate of her position is decreasing. She is shooting arrows at a a bunch of malevolent baboons which are attacking her as she moves. Her movements are limited so that when her arrow leaves the bow it will follow the path of the tangent line to the curve (in the direction in which x is decreasing as shown below). An angry baboon is positioned at the point $B(0,0)$. What are the coordinates of the point K on the curve at which Katniss should release an arrow in order to hit this baboon?



Solution:
We need to find the point on the graph where the tangent line interesects $(0,0)$. Let $f(x) = x^2 + 2$, and let $K = (x_0, y_0)$. The slope of the tangent line at x_0 is $f'(x_0)$, and an equation for the tangent line at this point is given by:

$$y - y_0 = f'(x_0)(x - x_0)$$

Note that we want both $(0,0)$ and $(x_0, y_0) = (x_0, x_0^2 + 2)$ to lie on this line, hence:

$$0 - (x_0^2 + 2) = f'(x_0)(0 - x_0).$$

As

$$f'(x_0) = 2x_0,$$

this means that

$$x_0^2 + 2 = 2x_0^2 \Longrightarrow x_0^2 - 2 = 0 \Longrightarrow (x_0 + \sqrt{2})(x_0 - \sqrt{2}) = 0.$$

In other words, $(\sqrt{2}, 4)$ is our desired point. (Note: Katniss is moving to the left, so she is not facing the right direction at $x_0 = -\sqrt{2}$)

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Rough Work